Chapter 3

Sequences

In this chapter, we discuss sequences. We say what it means for a sequence to converge, and define the limit of a convergent sequence. We begin with some preliminary results about the absolute value, which can be used to define a distance function, or metric, on $\mathbb{R}$. In turn, convergence is defined in terms of this metric.

3.1. The absolute value

Definition 3.1. The absolute value of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Some basic properties of the absolute value are the following.

Proposition 3.2. For all $x, y \in \mathbb{R}$:

(a) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;

(b) $|-x| = |x|$;

(c) $|x + y| \leq |x| + |y|$ (triangle inequality);

(d) $|xy| = |x|\,|y|$;

Proof. Parts (a), (b) follow immediately from the definition. Part (c) remains valid if we change the signs of both x and y or exchange x and y. Therefore we can assume that $x \geq 0$ and $|x| \geq |y|$ without loss of generality, in which case $x + y \geq 0$. If $y \geq 0$, corresponding to the case when x and y have the same sign, then

$$|x + y| = x + y = |x| + |y|.$$

If $y < 0$, corresponding to the case when x and y have opposite signs and $x+y > 0$, then

$$|x + y| = x + y = |x| - |y| < |x| + |y|,$$

which proves (c). Part (d) remains valid if we change x to $-x$ or y to $-y$, so we can assume that $x, y \geq 0$ without loss of generality. Then $xy \geq 0$ and $|xy| = xy = |x||y|$. □

One useful consequence of the triangle inequality is the following reverse triangle inequality.

Proposition 3.3. If $x, y \in \mathbb{R}$, then

$$||x| - |y|| \leq |x - y|.$$

Proof. By the triangle inequality,

$$|x| = |x - y + y| \leq |x - y| + |y|$$

so $|x| - |y| \leq |x - y|$. Similarly, exchanging x and y, we get $|y| - |x| \leq |x - y|$, which proves the result. □

We can give an equivalent condition for the boundedness of a set by using the absolute value instead of upper and lower bounds as in Definition 2.9.

Proposition 3.4. A set $A \subset \mathbb{R}$ is bounded if and only if there exists a real number $M \geq 0$ such that

$$|x| \leq M \text{ for every } x \in A.$$

Proof. If the condition in the proposition holds, then M is an upper bound of A and $-M$ is a lower bound, so A is bounded. Conversely, if A is bounded from above by M' and from below by m', then $|x| \leq M$ for every $x \in A$ where $M = \max\{|m'|, |M'|\}$. □

A third way to say that a set is bounded is in terms of its diameter.

Definition 3.5. Let $A \subset \mathbb{R}$. The diameter of A is

$$\operatorname{diam} A = \sup \{|x - y| : x, y \in A\}.$$

Then a set is bounded if and only if its diameter is finite.

Example 3.6. If $A = (-a, a)$, then $\operatorname{diam} A = 2a$, and A is bounded. If $A = (-\infty, a)$, then $\operatorname{diam} A = \infty$, and A is unbounded.

3.2. Sequences

A sequence (x_n) of real numbers is an ordered list of numbers $x_n \in \mathbb{R}$, called the terms of the sequence, indexed by the natural numbers $n \in \mathbb{N}$. We often indicate a sequence by listing the first few terms, especially if they have an obvious pattern. Of course, no finite list of terms is, on its own, sufficient to define a sequence.

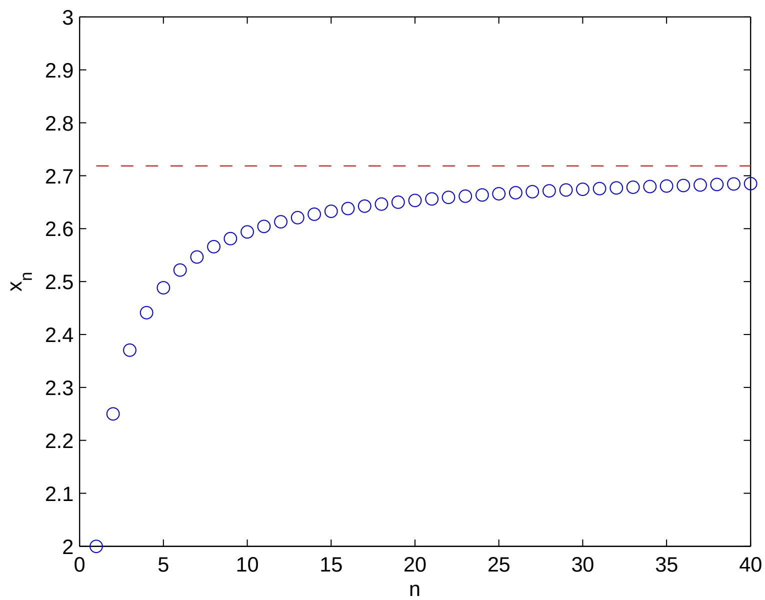

Figure 1. A plot of the first 40 terms in the sequence $x_n = (1 + 1/n)^n$, illustrating that it is monotone increasing and converges to $e \approx 2.718$, whose value is indicated by the dashed line.

Example 3.7. Here are some sequences:

$$\begin{aligned} 1,\ 8,\ 27, 64,\ \dots, \qquad & x_n = n^3, \\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots \qquad & x_n = \frac{1}{n}; \\ 1,\ -1,\ 1,\ -1,\ \dots \qquad & x_n = (-1)^{n+1}, \\ (1+1),\ \left(1+\frac{1}{2}\right)^2,\ \left(1+\frac{1}{3}\right)^3,\ \dots \qquad & x_n = \left(1+\frac{1}{n}\right)^n. \end{aligned}$$

Note that unlike sets, where elements are not repeated, the terms in a sequence may be repeated.

The formal definition of a sequence is as a function on $\mathbb{N}$, which is equivalent to its definition as a list.

Definition 3.8. A sequence (x_n) of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$, where $x_n = f(n)$.

We can consider sequences of many different types of objects (for example, sequences of functions) but for now we only consider sequences of real numbers, and we will refer to them as sequences for short. A useful way to visualize a sequence (x_n) is to plot the graph of $x_n \in \mathbb{R}$ versus $n \in \mathbb{N}$. (See Figure 1 for an example.)

If we want to indicate the range of the index $n \in \mathbb{N}$ explicitly, we write the sequence as $(x_n)_{n=1}^{\infty}$. Sometimes it is convenient to start numbering a sequence from a different integer, such as $n = 0$ instead of $n = 1$. In that case, a sequence $(x_n)_{n=0}^{\infty}$ is a function $f : \mathbb{N}_0 \to \mathbb{R}$ where $x_n = f(n)$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, and similarly for other starting points.

Every function $f : \mathbb{N} \to \mathbb{R}$ defines a sequence, corresponding to an arbitrary choice of a real number $x_n \in \mathbb{R}$ for each $n \in \mathbb{N}$. Some sequences can be defined explicitly by giving an expression for the nth terms, as in Example 3.7; others can be defined recursively. That is, we specify the value of the initial term (or terms) in the sequence, and define x_n as a function of the previous terms $(x_1, x_2, \dots, x_{n-1})$.

A well-known example of a recursive sequence is the Fibonacci sequence (F_n)

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \dots,$$

which is defined by $F_1 = F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 3.$$

That is, we add the two preceding terms to get the next term. In general, we cannot expect to solve a recursion relation to get an explicit expression for the nth term in a sequence, but the recursion relation for the Fibonacci sequence is linear with constant coefficients, and it can be solved to give an expression for F_n called the Euler-Binet formula.

Proposition 3.9 (Euler-Binet formula)**.** The nth term in the Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right], \qquad \phi = \frac{1+\sqrt{5}}{2}.$$

Proof. The terms in the Fibonacci sequence are uniquely determined by the linear difference equation

$$F_n - F_{n-1} - F_{n-2} = 0, \qquad n \geq 3,$$

with the initial conditions

$$F_1 = 1, \qquad F_2 = 1.$$

We see that $F_n = r^n$ is a solution of the difference equation if r satisfies

$$r^2 - r - 1 = 0,$$

which gives

$$r = \phi \text{ or } -\frac{1}{\phi}, \qquad \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803.$$

By linearity,

$$F_n = A\phi^n + B\left(-\frac{1}{\phi}\right)^n$$

is a solution of the difference equation for arbitrary constants A, B. This solution satisfies the initial conditions $F_1 = F_2 = 1$ if

$$A = \frac{1}{\sqrt{5}}, \qquad B = -\frac{1}{\sqrt{5}},$$

which proves the result. □

Alternatively, once we know the answer, we can prove Proposition 3.9 by induction. The details are left as an exercise. Note that although the right-hand side of the equation for F_n involves the irrational number $\sqrt{5}$, its value is an integer for every $n \in \mathbb{N}$.

The number ϕ appearing in Proposition 3.9 is called the golden ratio. It has the property that subtracting 1 from it gives its reciprocal, or

$$\phi - 1 = \frac{1}{\phi}.$$

Geometrically, this property means that the removal of a square from a rectangle whose sides are in the ratio ϕ leaves a rectangle whose sides are in the same ratio. The number ϕ was originally defined in Euclid's Elements as the division of a line in "extreme and mean ratio," and Ancient Greek architects arguably used rectangles with this proportion in the Parthenon and other buildings. During the Renaissance, ϕ was referred to as the "divine proportion." The first use of the term "golden section" appears to be by Martin Ohm, brother of the physicist Georg Ohm, in a book published in 1835.

3.3. Convergence and limits

Roughly speaking, a sequence (x_n) converges to a limit x if its terms x_n get arbitrarily close to x for all sufficiently large n.

Definition 3.10. A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$, written

$$x = \lim_{n\to\infty} x_n, \quad \text{or} \quad x_n \to x \text{ as } n \to \infty,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \qquad \text{for all } n > N.$$

A sequence converges if it converges to some limit $x \in \mathbb{R}$, otherwise it diverges.

Although we don't show it explicitly in the definition, N is allowed to depend on ϵ. Typically, the smaller we choose ϵ, the larger we have to make N. One way to view a proof of convergence is as a game: If I give you an $\epsilon > 0$, you have to come up with an N that "works." Also note that $x_n \to x$ as $n \to \infty$ means the same thing as $|x_n - x| \to 0$ as $n \to \infty$.

It may appear obvious that a limit is unique if one exists, but this fact requires proof.

Proposition 3.11. If a sequence converges, then its limit is unique.

Proof. Suppose that (x_n) is a sequence such that $x_n \to x$ and $x_n \to x'$ as $n \to \infty$. Let $\epsilon > 0$. Then there exist $N, N' \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2} \qquad \text{for all } n > N,$$

$$|x_n - x'| < \frac{\epsilon}{2} \qquad \text{for all } n > N'.$$

Choose any $n > \max\{N, N'\}$. Then, by the triangle inequality,

$$|x - x'| \le |x - x_n| + |x_n - x'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Since this inequality holds for all $\epsilon > 0$, we must have $|x - x'| = 0$ (otherwise the inequality would be false for $\epsilon = |x - x'|/2 > 0$), so $x = x'$. □

The following notation for sequences that "diverge to infinity" is convenient.

Definition 3.12. If (x_n) is a sequence then

$$\lim_{n\to\infty} x_n = \infty,$$

or $x_n \to \infty$ as $n \to \infty$, if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that

$$x_n > M \qquad \text{for all } n > N.$$

Also

$$\lim_{n\to\infty} x_n = -\infty,$$

or $x_n \to -\infty$ as $n \to \infty$, if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that

$$x_n < M \qquad \text{for all } n > N.$$

That is, $x_n \to \infty$ as $n \to \infty$ means the terms of the sequence (x_n) get arbitrarily large and positive for all sufficiently large n, while $x_n \to -\infty$ as $n \to \infty$ means that the terms get arbitrarily large and negative for all sufficiently large n. The notation $x_n \to \pm\infty$ does *not* mean that the sequence converges.

To illustrate these definitions, we discuss the convergence of the sequences in Example 3.7.

Example 3.13. The terms in the sequence

$$1,\ 8,\ 27,\ 64,\ \dots, \qquad x_n = n^3$$

eventually exceed any real number, so $n^3 \to \infty$ as $n \to \infty$ and this sequence does not converge. Explicitly, let $M \in \mathbb{R}$ be given, and choose $N \in \mathbb{N}$ such that $N > M^{1/3}$. (If $-\infty < M < 1$, we can choose $N = 1$.) Then for all $n > N$, we have $n^3 > N^3 > M$, which proves the result.

Example 3.14. The terms in the sequence

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots \qquad x_n = \frac{1}{n}$$

get closer to 0 as n gets larger, and the sequence converges to 0:

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

To prove this limit, let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. (Such a number exists by the Archimedean property of $\mathbb{R}$ stated in Theorem 2.18.) Then for all $n > N$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \epsilon,$$

which proves that $1/n \to 0$ as $n \to \infty$.

Example 3.15. The terms in the sequence

$$1,\ -1,\ 1,\ -1,\ \ldots \qquad x_n = (-1)^{n+1},$$

oscillate back and forth infinitely often between 1 and -1, but they do not approach any fixed limit, so the sequence does not converge. To show this explicitly, note that for every $x \in \mathbb{R}$ we have either $|x-1| \geq 1$ or $|x+1| \geq 1$. It follows that there is no $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n > N$. Thus, Definition 3.10 fails if $\epsilon = 1$ however we choose $x \in \mathbb{R}$, and the sequence does not converge.

Example 3.16. The convergence of the sequence

$$(1+1),\ \left(1+\frac{1}{2}\right)^2,\ \left(1+\frac{1}{3}\right)^3,\ldots \qquad x_n = \left(1+\frac{1}{n}\right)^n,$$

illustrated in Figure 1, is less obvious. Its terms are given by

$$x_n = a_n^n, \qquad a_n = 1 + \frac{1}{n}.$$

As n increases, we take larger powers of numbers that get closer to one. If $a > 1$ is any fixed real number, then $a^n \to \infty$ as $n \to \infty$ so the sequence (a^n) does not converge (see Proposition 3.31 below for a detailed proof). On the other hand, if $a = 1$, then $1^n = 1$ for all $n \in \mathbb{N}$ so the sequence (1^n) converges to 1. Thus, there are two competing factors in the sequence with increasing n: $a_n \to 1$ but $n \to \infty$. It is not immediately obvious which of these factors "wins."

In fact, they are in balance. As we prove in Proposition 3.32 below, the sequence converges with

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e,$$

where $2 < e < 3$. This limit can be taken as the definition of $e \approx 2.71828$.

For comparison, one can also show that

$$\lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n = 1, \qquad \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n^2} = \infty.$$

In the first case, the rapid approach of $a_n = 1+1/n^2$ to 1 "beats" the slower growth in the exponent n, while in the second case, the rapid growth in the exponent n^2 "beats" the slower approach of $a_n = 1 + 1/n$ to 1.

An important property of a sequence is whether or not it is bounded.

Definition 3.17. A sequence (x_n) of real numbers is bounded from above if there exists $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$, and bounded from below if there exists $m \in \mathbb{R}$ such that $x_n \geq m$ for all $n \in \mathbb{N}$. A sequence is bounded if it is bounded from above and below, otherwise it is unbounded.

An equivalent condition for a sequence (x_n) to be bounded is that there exists $M \geq 0$ such that

$$|x_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Example 3.18. The sequence (n^3) is bounded from below but not from above, while the sequences $(1/n)$ and $\big((-1)^{n+1}\big)$ are bounded. The sequence

$$1,\ -2,\ 3,\ -4,\ 5,\ -6,\ \dots \qquad x_n = (-1)^{n+1} n$$

is not bounded from below or above.

We then have the following property of convergent sequences.

Proposition 3.19. A convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit x. There exists $N \in \mathbb{N}$ such that

$$|x_n - x| < 1 \qquad \text{for all } n > N.$$

The triangle inequality implies that

$$|x_n| \le |x_n - x| + |x| < 1 + |x| \qquad \text{for all } n > N.$$

Defining

$$M = \max\{|x_1|, |x_2|, \dots, |x_N|, 1 + |x|\},$$

we see that $|x_n| \le M$ for all $n \in \mathbb{N}$, so (x_n) is bounded. □

Thus, boundedness is a necessary condition for convergence, and every unbounded sequence diverges; for example, the unbounded sequence in Example 3.13 diverges. On the other hand, boundedness is not a sufficient condition for convergence; for example, the bounded sequence in Example 3.15 diverges.

The boundedness, or convergence, of a sequence $(x_n)_{n=1}^{\infty}$ depends only on the behavior of the infinite "tails" $(x_n)_{n=N}^{\infty}$ of the sequence, where N is arbitrarily large. Equivalently, the sequence $(x_n)_{n=1}^{\infty}$ and the shifted sequences $(x_{n+N})_{n=1}^{\infty}$ have the same convergence properties and limits for every $N \in \mathbb{N}$. As a result, changing a finite number of terms in a sequence doesn't alter its boundedness or convergence, nor does it alter the limit of a convergent sequence. In particular, the existence of a limit gives no information about how quickly a sequence converges to its limit.

Example 3.20. Changing the first hundred terms of the sequence $(1/n)$ from $1/n$ to n, we get the sequence

$$1,\ 2,\ 3,\ \dots,\ 99,\ 100,\ \frac{1}{101},\ \frac{1}{102},\ \frac{1}{103},\ \dots,$$

which is still bounded (although by 100 instead of by 1) and still convergent to 0. Similarly, changing the first billion terms in the sequence doesn't change its boundedness or convergence.

We introduce some convenient terminology to describe the behavior of "tails" of a sequence,

Definition 3.21. Let $P(x)$ denote a property of real numbers $x \in \mathbb{R}$. If (x_n) is a real sequence, then $P(x_n)$ holds eventually if there exists $N \in \mathbb{N}$ such that $P(x_n)$ holds for all $n > N$; and $P(x_n)$ holds infinitely often if for every $N \in \mathbb{N}$ there exists $n > N$ such that $P(x_n)$ holds.

For example, (x_n) is bounded if there exists $M \geq 0$ such that $|x_n| \leq M$ eventually; and (x_n) does not converge to $x \in \mathbb{R}$ if there exists $\epsilon_0 > 0$ such that $|x_n - x| \geq \epsilon_0$ infinitely often.

Note that if a property P holds infinitely often according to Definition 3.21, then it does indeed hold infinitely often: If $N = 1$, then there exists $n_1 > 1$ such that $P(x_{n_1})$ holds; if $N = n_1$, then there exists $n_2 > n_1$ such that $P(x_{n_2})$ holds; then there exists $n_3 > n_2$ such that $P(x_{n_3})$ holds, and so on.

3.4. Properties of limits

In this section, we prove some order and algebraic properties of limits of sequences.

3.4.1. Monotonicity. Limits of convergent sequences preserve (non-strict) inequalities.

Theorem 3.22. If (x_n) and (y_n) are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then for every $\epsilon > 0$ there exists $P, Q \in \mathbb{N}$ such that

$$|x - x_n| < \frac{\epsilon}{2} \qquad \text{for all } n > P,$$
$$|y - y_n| < \frac{\epsilon}{2} \qquad \text{for all } n > Q.$$

Choosing $n > \max\{P, Q\}$, we have

$$x = x_n + x - x_n < y_n + \frac{\epsilon}{2} = y + y_n - y + \frac{\epsilon}{2} < y + \epsilon.$$

Since $x < y + \epsilon$ for every $\epsilon > 0$, it follows that $x \leq y$. □

This result, of course, remains valid if the inequality $x_n \leq y_n$ holds only for all sufficiently large n. Limits need not preserve strict inequalities. For example, $1/n > 0$ for all $n \in \mathbb{N}$ but $\lim_{n\to\infty} 1/n = 0$.

It follows immediately that if (x_n) is a convergent sequence with $m \leq x_n \leq M$ for all $n \in \mathbb{N}$, then

$$m \leq \lim_{n\to\infty} x_n \leq M.$$

The following "squeeze" or "sandwich" theorem is often useful in proving the convergence of a sequence by bounding it between two simpler convergent sequences with equal limits.

Theorem 3.23 (Sandwich)**.** Suppose that (x_n) and (y_n) are convergent sequences of real numbers with the same limit L. If (z_n) is a sequence such that

$$x_n \leq z_n \leq y_n \qquad \text{for all } n \in \mathbb{N},$$

then (z_n) also converges to L.

Proof. Let $\epsilon > 0$ be given, and choose $P, Q \in \mathbb{N}$ such that

$$|x_n - L| < \epsilon \quad \text{for all } n > P, \qquad |y_n - L| < \epsilon \quad \text{for all } n > Q.$$

If $N = \max\{P, Q\}$, then for all $n > N$

$$-\epsilon < x_n - L \le z_n - L \le y_n - L < \epsilon,$$

which implies that $|z_n - L| < \epsilon$. This prove the result. □

It is essential here that (x_n) and (y_n) have the same limit.

Example 3.24. If $x_n = -1$, $y_n = 1$, and $z_n = (-1)^{n+1}$, then $x_n \le z_n \le y_n$ for all $n \in \mathbb{N}$, the sequence (x_n) converges to -1 and (y_n) converges 1, but (z_n) does not converge.

As once consequence, we show that we can take absolute values inside limits.

Corollary 3.25. If $x_n \to x$ as $n \to \infty$, then $|x_n| \to |x|$ as $n \to \infty$.

Proof. By the reverse triangle inequality,

$$0 \le \big|\, |x_n| - |x| \,\big| \le |x_n - x|,$$

and the result follows from Theorem 3.23. □

3.4.2. Linearity. Limits respect addition and multiplication. In proving the following theorem, we need to show that the sequences converge, not just get an expressions for their limits.

Theorem 3.26. Suppose that (x_n) and (y_n) are convergent real sequences and $c \in \mathbb{R}$. Then the sequences (cx_n), $(x_n + y_n)$, and $(x_n y_n)$ converge, and

$$\lim_{n\to\infty} cx_n = c \lim_{n\to\infty} x_n,$$
$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$
$$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

Proof. We let

$$x = \lim_{n\to\infty} x_n, \qquad y = \lim_{n\to\infty} y_n.$$

The first statement is immediate if $c = 0$. Otherwise, let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{|c|} \qquad \text{for all } n > N.$$

Then

$$|cx_n - cx| < \epsilon \qquad \text{for all } n > N,$$

which proves that (cx_n) converges to cx.

For the second statement, let $\epsilon > 0$ be given, and choose $P, Q \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2} \quad \text{for all } n > P, \qquad |y_n - y| < \frac{\epsilon}{2} \quad \text{for all } n > Q.$$

Let $N = \max\{P, Q\}$. Then for all $n > N$, we have

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| < \epsilon,$$

which proves that $(x_n + y_n)$ converges to $x + y$.

For the third statement, note that since (x_n) and (y_n) converge, they are bounded and there exists $M > 0$ such that

$$|x_n|, |y_n| \leq M \qquad \text{for all } n \in \mathbb{N}$$

and $|x|, |y| \leq M$. Given $\epsilon > 0$, choose $P, Q \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2M} \quad \text{for all } n > P, \qquad |y_n - y| < \frac{\epsilon}{2M} \quad \text{for all } n > Q,$$

and let $N = \max\{P, Q\}$. Then for all $n > N$,

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + x(y_n - y)| \\ &\leq |x_n - x|\,|y_n| + |x|\,|y_n - y| \\ &\leq M\,(|x_n - x| + |y_n - y|) \\ &< \epsilon, \end{aligned}$$

which proves that $(x_n y_n)$ converges to xy. □

Note that the convergence of $(x_n + y_n)$ does not imply the convergence of (x_n) and (y_n) separately; for example, take $x_n = n$ and $y_n = -n$. If, however, (x_n) converges then (y_n) converges if and only if $(x_n + y_n)$ converges.

3.5. Monotone sequences

Monotone sequences have particularly simple convergence properties.

Definition 3.27. A sequence (x_n) of real numbers is increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$, decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, and monotone if it is increasing or decreasing. A sequence is strictly increasing if $x_{n+1} > x_n$, strictly decreasing if $x_{n+1} < x_n$, and strictly monotone if it is strictly increasing or strictly decreasing.

We don't require a monotone sequence to be strictly monotone, but this usage isn't universal. In some places, "increasing" or "decreasing" is used to mean "strictly increasing" or "strictly decreasing." In that case, what we call an increasing sequence is called a nondecreasing sequence and a decreasing sequence is called nonincreasing sequence. We'll use the more easily understood direct terminology.

Example 3.28. The sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ \dots$$

is monotone increasing but not strictly monotone increasing; the sequence (n^3) is strictly monotone increasing; the sequence $(1/n)$ is strictly monotone decreasing; and the sequence $((-1)^{n+1})$ is not monotone.

Bounded monotone sequences always converge, and unbounded monotone sequences diverge to $\pm\infty$.

Theorem 3.29. A monotone sequence of real numbers converges if and only if it is bounded. If (x_n) is monotone increasing and bounded, then

$$\lim_{n \to \infty} x_n = \sup\{x_n : n \in \mathbb{N}\},$$

and if (x_n) is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Furthermore, if (x_n) is monotone increasing and unbounded, then

$$\lim_{n\to\infty} x_n = \infty,$$

and if (x_n) is monotone decreasing and unbounded, then

$$\lim_{n\to\infty} x_n = -\infty.$$

Proof. If the sequence converges, then by Proposition 3.19 it is bounded.

Conversely, suppose that (x_n) is a bounded, monotone increasing sequence. The set of terms $\{x_n : n \in \mathbb{N}\}$ is bounded from above, so by Axiom 2.17 it has a supremum

$$x = \sup\{x_n : n \in \mathbb{N}\}.$$

Let $\epsilon > 0$. From the definition of the supremum, there exists an $N \in \mathbb{N}$ such that $x_N > x - \epsilon$. Since the sequence is increasing, we have $x_n \geq x_N$ for all $n > N$, and therefore $x - \epsilon < x_n \leq x$. It follows that

$$|x_n - x| < \epsilon \qquad \text{for all } n > N,$$

which proves that $x_n \to x$ as $n \to \infty$.

If (x_n) is an unbounded monotone increasing sequence, then it is not bounded from above, since it is bounded from below by its first term x_1. Hence, for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_N > M$. Since the sequence is increasing, we have $x_n \geq x_N > M$ for all $n > N$, which proves that $x_n \to \infty$ as $n \to \infty$.

The result for a monotone decreasing sequence (x_n) follows similarly, or by applying the previous result to the monotone increasing sequence $(-x_n)$. □

The fact that every bounded monotone sequence has a limit is another way to express the completeness of $\mathbb{R}$. For example, this is not true in $\mathbb{Q}$: an increasing sequence of rational numbers that converges to $\sqrt{2}$ is bounded from above in $\mathbb{Q}$ (for example, by 2) but has no limit in $\mathbb{Q}$.

We sometimes use the notation $x_n \uparrow x$ to indicate that (x_n) is a monotone increasing sequence that converges to x, and $x_n \downarrow x$ to indicate that (x_n) is a monotone decreasing sequence that converges to x, with a similar notation for monotone sequences that diverge to $\pm\infty$. For example, $1/n \downarrow 0$ and $n^3 \uparrow \infty$ as $n \to \infty$.

The following propositions give some examples of monotone sequences. In the proofs, we use the binomial theorem, which we state without proof.

Theorem 3.30 (Binomial)**.** If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Here, $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ and, by convention, $0! = 1$. The binomial coefficients

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot k},$$

read "n choose k," give the number of ways of choosing k objects from n objects, order not counting. For example,

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2,\\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3,\\ (x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

We also recall the sum of a geometric series: if $a \neq 1$, then

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

Proposition 3.31. The geometric sequence $(a^n)_{n=0}^{\infty}$,

$$1,\ a,\ a^2,\ a^3,\ \dots,$$

is strictly monotone decreasing if $0 < a < 1$, with

$$\lim_{n\to\infty} a^n = 0,$$

and strictly monotone increasing if $1 < a < \infty$, with

$$\lim_{n\to\infty} a^n = \infty.$$

Proof. If $0 < a < 1$, then $0 < a^{n+1} = a \cdot a^n < a^n$, so the sequence (a^n) is strictly monotone decreasing and bounded from below by zero. Therefore by Theorem 3.29 it has a limit $x \in \mathbb{R}$. Theorem 3.26 implies that

$$x = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a \cdot a^n = a \lim_{n\to\infty} a^n = ax.$$

Since $a \neq 1$, it follows that $x = 0$.

If $a > 1$, then $a^{n+1} = a \cdot a^n > a^n$, so (a^n) is strictly increasing. Let $a = 1 + \delta$ where $\delta > 0$. By the binomial theorem, we have

$$\begin{aligned}a^n &= (1+\delta)^n\\ &= \sum_{k=0}^{n} \binom{n}{k} \delta^k\\ &= 1 + n\delta + \frac{1}{2}n(n-1)\delta^2 + \dots + \delta^n\\ &> 1 + n\delta.\end{aligned}$$

Given $M \geq 0$, choose $N \in \mathbb{N}$ such that $N > M/\delta$. Then for all $n > N$, we have

$$a^n > 1 + n\delta > 1 + N\delta > M,$$

so $a_n \to \infty$ as $n \to \infty$. □

The next proposition proves the existence of the limit for e in Example 3.16.

Proposition 3.32. The sequence (x_n) with

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is strictly monotone increasing and converges to a limit $2 < e < 3$.

Proof. By the binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\\ &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}\\ &\qquad +\dots+\frac{n(n-1)(n-2)\dots 2\cdot 1}{n!}\cdot\frac{1}{n^n}\\ &= 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\\ &\qquad +\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\frac{2}{n}\cdot\frac{1}{n}.\end{aligned}$$

Each of the terms in the sum on the right hand side is a positive increasing function of n, and the number of terms increases with n. Therefore (x_n) is a strictly increasing sequence, and $x_n > 2$ for every $n \geq 2$. Moreover, since $0 \leq (1-k/n) < 1$ for $1 \leq k \leq n$, we have

$$\left(1+\frac{1}{n}\right)^n < 2+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}.$$

Since $n! \geq 2^{n-1}$ for $n \geq 1$, it follows that

$$\left(1+\frac{1}{n}\right)^n < 2+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}} = 2+\frac{1}{2}\left[\frac{1-(1/2)^{n-1}}{1-1/2}\right] < 3,$$

so (x_n) is monotone increasing and bounded from above by a number strictly less than 3. By Theorem 3.29 the sequence converges to a limit $2 < e < 3$. □

3.6. The lim sup and lim inf

The lim sup and lim inf allow us to reduce questions about the convergence and limits of general real sequences to ones about monotone sequences. They are somewhat subtle concepts, and after defining them we will consider a number of examples.

Unlike the limit, the lim sup and lim inf of every bounded sequence of real numbers exist. A sequence converges if and only if its lim sup and lim inf are equal, in which case its limit is their common value. Furthermore, a sequence is unbounded if and only if at least one of its lim sup or lim inf diverges to $\pm\infty$, and it diverges to $\pm\infty$ if and only if both its lim sup and lim inf diverge to $\pm\infty$.

In order to define the lim sup and lim inf of a sequence (x_n) of real numbers, we introduce the sequences (y_n) and (z_n) obtained by taking the supremum and infimum, respectively, of the "tails" of (x_n):

$$y_n = \sup\{x_k : k \geq n\}, \qquad z_n = \inf\{x_k : k \geq n\}.$$

As n increases, the supremum and infimum are taken over smaller sets, so (y_n) is monotone decreasing and (z_n) is monotone increasing. The limits of these sequences are the lim sup and lim inf, respectively, of the original sequence.

Definition 3.33. Suppose that (x_n) is a sequence of real numbers. Then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n, \qquad y_n = \sup\left\{x_k : k \geq n\right\},$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} z_n, \qquad z_n = \inf\left\{x_k : k \geq n\right\}.$$

Note that $\limsup x_n$ exists and is finite provided that each of the y_n is finite and (y_n) is bounded from below; similarly, $\liminf x_n$ exists and is finite provided that each of the z_n is finite and (z_n) is bounded from above. We may also write

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} x_k\right), \qquad \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} x_k\right).$$

As for the limit of monotone sequences, it is convenient to allow the lim inf or lim sup to be $\pm\infty$, and we state explicitly what this means. We have $-\infty < y_n \leq \infty$ for every $n \in \mathbb{N}$, since the supremum of a non-empty set cannot be $-\infty$, but we may have $y_n \downarrow -\infty$; similarly, $-\infty \leq z_n < \infty$, but we may have $z_n \uparrow \infty$. These possibilities lead to the following cases.

Definition 3.34. Suppose that (x_n) is a sequence of real numbers and the sequences (y_n), (z_n) of possibly extended real numbers are given by Definition 3.33. Then

$$\limsup_{n\to\infty} x_n = \infty \qquad \text{if } y_n = \infty \text{ for every } n \in \mathbb{N},$$
$$\limsup_{n\to\infty} x_n = -\infty \qquad \text{if } y_n \downarrow -\infty \text{ as } n \to \infty,$$
$$\liminf_{n\to\infty} x_n = -\infty \qquad \text{if } z_n = -\infty \text{ for every } n \in \mathbb{N},$$
$$\liminf_{n\to\infty} x_n = \infty \qquad \text{if } z_n \uparrow \infty \text{ as } n \to \infty.$$

In all cases, we have $z_n \leq y_n$ for every $n \in \mathbb{N}$, with the usual ordering conventions for $\pm\infty$, and by taking the limit as $n \to \infty$, we get that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

We illustrate the definition of the lim sup and lim inf with a number of examples.

Example 3.35. Consider the bounded, increasing sequence

$$0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4}, \ldots \qquad x_n = 1 - \frac{1}{n}.$$

Defining y_n and z_n as above, we have

$$y_n = \sup\left\{1 - \frac{1}{k} : k \geq n\right\} = 1, \quad z_n = \inf\left\{1 - \frac{1}{k} : k \geq n\right\} = 1 - \frac{1}{n},$$

and both $y_n \downarrow 1$ and $z_n \uparrow 1$ converge monotonically to the limit 1 of the original sequence. Thus,

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = 1.$$

Example 3.36. Consider the bounded, non-monotone sequence

$$1,\ -1,\ 1,\ -1, \ldots \qquad x_n = (-1)^{n+1}.$$

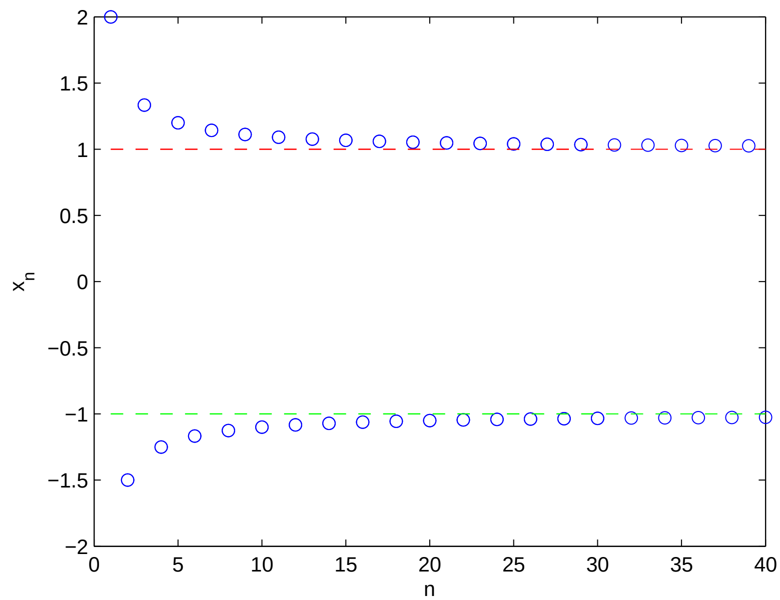

Figure 2. A plot of the first 40 terms in the sequence $x_n = (-1)^{n+1}(1+1/n)$ in Example 3.37. The dashed lines show $\limsup x_n = 1$ and $\liminf x_n = -1$.

Then

$$y_n = \sup\left\{(-1)^{k+1} : k \geq n\right\} = 1, \quad z_n = \inf\left\{(-1)^{k+1} : k \geq n\right\} = -1,$$

and $y_n \downarrow 1$, $z_n \uparrow -1$ converge to different limits. Thus,

$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1.$$

The original sequence does not converge, and $\lim x_n$ is undefined.

Example 3.37. The bounded, non-monotone sequence

$$2,\ -\frac{3}{2},\ \frac{4}{3},\ -\frac{5}{4},\dots \qquad x_n = (-1)^{n+1}\left(1+\frac{1}{n}\right)$$

is shown in Figure 2. We have

$$y_n = \sup\{x_k : k \geq n\} = \begin{cases} 1 + 1/n & \text{if } n \text{ is odd,} \\ 1 + 1/(n+1) & \text{if } n \text{ is even,} \end{cases}$$

$$z_n = \inf\{x_k : k \geq n\} = \begin{cases} -[1 + 1/(n+1)] & \text{if } n \text{ is odd,} \\ -[1 + 1/n] & \text{if } n \text{ is even,} \end{cases}$$

and it follows that

$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1.$$

The limit of the sequence does not exist. Note that infinitely many terms of the sequence are strictly greater than $\limsup x_n$, so $\limsup x_n$ does not bound any "tail" of the sequence from above. However, every number strictly greater than $\limsup x_n$ eventually bounds the sequence from above. Similarly, $\liminf x_n$ does not bound any "tail" of the sequence from below, but every number strictly less than $\liminf x_n$ eventually bounds the sequence from below.

Example 3.38. Consider the unbounded, increasing sequence

$$1,\ 2,\ 3,\ 4,\ 5, \dots \qquad x_n = n.$$

We have

$$y_n = \sup\{x_k : k \geq n\} = \infty, \qquad z_n = \inf\{x_k : k \geq n\} = n,$$

so the lim sup, lim inf and lim all diverge to ∞,

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \infty.$$

Example 3.39. Consider the unbounded, non-monotone sequence

$$1,\ -2,\ 3,\ -4,\ 5, \dots \qquad x_n = \begin{cases} n & \text{if } n \text{ is odd,} \\ -n & \text{if } n \text{ is even.} \end{cases}$$

We have $y_n = \infty$ and $z_n = -\infty$ for every $n \in \mathbb{N}$, and

$$\limsup_{n\to\infty} x_n = \infty, \qquad \liminf_{n\to\infty} x_n = -\infty.$$

The sequence oscillates and does not diverge to either ∞ or $-\infty$, so $\lim x_n$ is undefined even as an extended real number.

Example 3.40. Consider the unbounded non-monotone sequence

$$1,\ \frac{1}{2},\ 3,\ \frac{1}{4},\ 5, \dots \qquad x_n = \begin{cases} n & \text{if } n \text{ is odd.} \\ 1/n & \text{if } n \text{ is even,} \end{cases}$$

Then $y_n = \infty$ and

$$z_n = \begin{cases} 1/n & \text{if } n \text{ even,} \\ 1/(n+1) & \text{if } n \text{ odd.} \end{cases}$$

Therefore

$$\limsup_{n\to\infty} x_n = \infty, \qquad \liminf_{n\to\infty} x_n = 0.$$

As noted above, the lim sup of a sequence needn't bound any tail of the sequence, but the sequence is eventually bounded from above by every number that is strictly greater than the lim sup, and the sequence is greater infinitely often than every number that is strictly less than the lim sup. This property gives an alternative characterization of the lim sup, one that we often use in practice.

Theorem 3.41. Let (x_n) be a real sequence. Then

$$y = \limsup_{n\to\infty} x_n$$

if and only if $-\infty \leq y \leq \infty$ satisfies one of the following conditions.

(1) $-\infty < y < \infty$ and for every $\epsilon > 0$: (a) there exists $N \in \mathbb{N}$ such that $x_n < y + \epsilon$ for all $n > N$; (b) for every $N \in \mathbb{N}$ there exists $n > N$ such that $x_n > y - \epsilon$.

(2) $y = \infty$ and for every $M \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x_n > M$, i.e., (x_n) is not bounded from above.

(3) $y = -\infty$ and for every $m \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n < m$ for all $n > N$, i.e., $x_n \to -\infty$ as $n \to \infty$.

Similarly,

$$z = \liminf_{n\to\infty} x_n$$

if and only if $-\infty \le z \le \infty$ satisfies one of the following conditions.

(1) $-\infty < z < \infty$ and for every $\epsilon > 0$: (a) there exists $N \in \mathbb{N}$ such that $x_n > z - \epsilon$ for all $n > N$; (b) for every $N \in \mathbb{N}$ there exists $n > N$ such that $x_n < z + \epsilon$.

(2) $z = -\infty$ and for every $m \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x_n < m$, i.e., (x_n) is not bounded from below.

(3) $z = \infty$ and for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n > N$, i.e., $x_n \to \infty$ as $n \to \infty$.

Proof. We prove the result for lim sup. The result for lim inf follows by applying this result to the sequence $(-x_n)$.

First, suppose that $y = \limsup x_n$ and $-\infty < y < \infty$. Then (x_n) is bounded from above and

$$y_n = \sup\{x_k : k \ge n\} \downarrow y \qquad \text{as } n \to \infty.$$

Therefore, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $y_N < y + \epsilon$. Since $x_n \le y_N$ for all $n > N$, this proves (1a). To prove (1b), let $\epsilon > 0$ and suppose that $N \in \mathbb{N}$ is arbitrary. Since $y_N \ge y$ is the supremum of $\{x_n : n \ge N\}$, there exists $n \ge N$ such that $x_n > y_N - \epsilon \ge y - \epsilon$, which proves (1b).

Conversely, suppose that $-\infty < y < \infty$ satisfies condition (1) for the lim sup. Then, given any $\epsilon > 0$, (1a) implies that there exists $N \in \mathbb{N}$ such that

$$y_n = \sup\{x_k : k \ge n\} \le y + \epsilon \qquad \text{for all } n > N,$$

and (1b) implies that $y_n > y - \epsilon$ for all $n \in \mathbb{N}$. Hence, $|y_n - y| < \epsilon$ for all $n > N$, so $y_n \to y$ as $n \to \infty$, which means that $y = \limsup x_n$.

We leave the verification of the equivalence for $y = \pm\infty$ as an exercise. □

Next we give a necessary and sufficient condition for the convergence of a sequence in terms of its lim inf and lim sup.

Theorem 3.42. A sequence (x_n) of real numbers converges if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$$

are finite and equal, in which case

$$\lim_{n\to\infty} x_n = x.$$

Furthermore, the sequence diverges to ∞ if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \infty$$

and diverges to $-\infty$ if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = -\infty$$

Proof. First suppose that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$$

for some $x \in \mathbb{R}$. Then $y_n \downarrow x$ and $z_n \uparrow x$ as $n \to \infty$ where

$$y_n = \sup\{x_k : k \geq n\}, \qquad z_n = \inf\{x_k : k \geq n\}.$$

Since $z_n \leq x_n \leq y_n$, the "sandwich" theorem implies that

$$\lim_{n\to\infty} x_n = x.$$

Conversely, suppose that the sequence (x_n) converges to a limit $x \in \mathbb{R}$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$x - \epsilon < x_n < x + \epsilon \qquad \text{for all } n > N.$$

It follows that

$$x - \epsilon \leq z_n \leq y_n \leq x + \epsilon \qquad \text{for all } n > N.$$

Therefore $y_n, z_n \to x$ as $n \to \infty$, so $\limsup x_n = \liminf x_n = x$.

The sequence (x_n) diverges to ∞ if and only if $\liminf x_n = \infty$, and then $\limsup x_n = \infty$, since $\liminf x_n \leq \limsup x_n$. Similarly, (x_n) diverges to $-\infty$ if and only if $\limsup x_n = -\infty$, and then $\liminf x_n = -\infty$. □

If $\liminf x_n \neq \limsup x_n$, then we say that the sequence (x_n) oscillates. The difference

$$\limsup x_n - \liminf x_n$$

provides a measure of the size of the oscillations in the sequence as $n \to \infty$.

Every sequence has a finite or infinite lim sup, but not every sequence has a limit (even if we include sequences that diverge to $\pm\infty$). The following corollary gives a convenient way to prove the convergence of a sequence without having to refer to the limit before it is known to exist.

Corollary 3.43. Let (x_n) be a sequence of real numbers. Then (x_n) converges with $\lim_{n\to\infty} x_n = x$ if and only if $\limsup_{n\to\infty} |x_n - x| = 0$.

Proof. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n - x| = 0$, so

$$\limsup_{n\to\infty} |x_n - x| = \lim_{n\to\infty} |x_n - x| = 0.$$

Conversely, if $\limsup_{n\to\infty} |x_n - x| = 0$, then

$$0 \leq \liminf_{n\to\infty} |x_n - x| \leq \limsup_{n\to\infty} |x_n - x| = 0,$$

so $\liminf_{n\to\infty} x_n|x_n - x| = \limsup_{n\to\infty} x_n|x_n - x| = 0$. Theorem 3.42 implies that $\lim_{n\to\infty} |x_n - x| = 0$, or $\lim_{n\to\infty} x_n = x$. □

Note that the condition $\liminf_{n\to\infty} |x_n - x| = 0$ doesn't tell us anything about the convergence of (x_n).

Example 3.44. Let $x_n = 1 + (-1)^n$. Then (x_n) oscillates between 0 and 2, and

$$\liminf_{n\to\infty} x_n = 0, \qquad \limsup_{n\to\infty} x_n = 2.$$

The sequence is non-negative and its lim inf is 0, but the sequence does not converge.

3.7. Cauchy sequences

> Cauchy has become unbearable. Every Monday, broadcasting the known facts he has learned over the week as a discovery. I believe there is no historical precedent for such a talent writing so much awful rubbish. This is why I have relegated him to the rank below us. (Jacobi in a letter to Dirichlet, 1841)

The Cauchy condition is a necessary and sufficient condition for the convergence of a real sequence that depends only on the terms of the sequence and not on its limit. Furthermore, the completeness of $\mathbb{R}$ can be defined by the convergence of Cauchy sequences, instead of by the existence of suprema. This approach defines completeness in terms of the distance properties of $\mathbb{R}$ rather than its order properties and generalizes to other metric spaces that don't have a natural ordering.

Roughly speaking, a Cauchy sequence is a sequence whose terms eventually get arbitrarily close together.

Definition 3.45. A sequence (x_n) of real numbers is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \epsilon \qquad \text{for all } m, n > N.$$

Theorem 3.46. A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. First suppose that (x_n) converges to a limit $x \in \mathbb{R}$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2} \qquad \text{for all } n > N.$$

It follows that if $m, n > N$, then

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \epsilon,$$

which implies that (x_n) is Cauchy. (This direction doesn't use the completeness of $\mathbb{R}$; for example, it holds equally well for sequence of rational numbers that converge in $\mathbb{Q}$.)

Conversely, suppose that (x_n) is Cauchy. Then there is $N_1 \in \mathbb{N}$ such that

$$|x_m - x_n| < 1 \qquad \text{for all } m, n > N_1.$$

It follows that if $n > N_1$, then

$$|x_n| \leq |x_n - x_{N_1+1}| + |x_{N_1+1}| \leq 1 + |x_{N_1+1}|.$$

Hence the sequence is bounded with

$$|x_n| \leq \max\left\{|x_1|, |x_2|, \dots, |x_{N_1}|, 1 + |x_{N_1+1}|\right\}.$$

Since the sequence is bounded, its lim sup and lim inf exist. We claim they are equal. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that the Cauchy condition in Definition 3.45 holds. Then

$$x_n - \epsilon < x_m < x_n + \epsilon \qquad \text{for all } m \geq n > N.$$

It follows that for all $n > N$ we have

$$x_n - \epsilon \leq \inf\left\{x_m : m \geq n\right\}, \quad \sup\left\{x_m : m \geq n\right\} \leq x_n + \epsilon,$$

which implies that

$$\sup\{x_m : m \geq n\} - \epsilon \leq \inf\{x_m : m \geq n\} + \epsilon.$$

Taking the limit as $n \to \infty$, we get that

$$\limsup_{n\to\infty} x_n - \epsilon \leq \liminf_{n\to\infty} x_n + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} x_n \leq \liminf_{n\to\infty} x_n.$$

It follows that $\limsup x_n = \liminf x_n$, so Theorem 3.42 implies that the sequence converges. □

3.8. Subsequences

A subsequence of a sequence (x_n)

$$x_1,\ x_2,\ x_3,\ \dots,\ x_n,\ \dots$$

is a sequence (x_{n_k}) of the form

$$x_{n_1},\ x_{n_2},\ x_{n_3},\ \dots,\ x_{n_k},\ \dots$$

where $n_1 < n_2 < n_3 < \dots < n_k < \dots$.

Example 3.47. A subsequence of the sequence $(1/n)$,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \dots.$$

is the sequence $(1/k^2)$

$$1,\ \frac{1}{4},\ \frac{1}{9},\ \frac{1}{16},\ \frac{1}{25},\ \dots.$$

Here, $n_k = k^2$. On the other hand, the sequences

$$1,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \dots, \qquad \frac{1}{2},\ 1,\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \dots$$

aren't subsequences of $(1/n)$ since n_k is not a strictly increasing function of k in either case.

The standard short-hand notation for subsequences used above is convenient but not entirely consistent, and the notion of a subsequence is a bit more involved than it might appear at first sight. To explain it in more detail, we give the formal definition of a subsequence as a function on $\mathbb{N}$.

Definition 3.48. Let (x_n) be a sequence, where $x_n = f(n)$ and $f : \mathbb{N} \to \mathbb{R}$. A sequence (y_k), where $y_k = g(k)$ and $g : \mathbb{N} \to \mathbb{R}$, is a subsequence of (x_n) if there is a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $g = f \circ \phi$. In that case, we write $\phi(k) = n_k$ and $y_k = x_{n_k}$.

Example 3.49. In Example 3.47, the sequence $(1/n)$ corresponds to the function $f(n) = 1/n$ and the subsequence $(1/k^2)$ corresponds to $g(k) = 1/k^2$. Here, $g = f \circ \phi$ with $\phi(k) = k^2$.

Note that since the indices in a subsequence form a strictly increasing sequence of integers (n_k), it follows that $n_k \to \infty$ as $k \to \infty$.

Proposition 3.50. Every subsequence of a convergent sequence converges to the limit of the sequence.

Proof. Suppose that (x_n) is a convergent sequence with $\lim_{n\to\infty} x_n = x$ and (x_{n_k}) is a subsequence. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n > N$. Since $n_k \to \infty$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $n_k > N$ if $k > K$. Then $k > K$ implies that $|x_{n_k} - x| < \epsilon$, so $\lim_{k\to\infty} x_{n_k} = x$. □

A useful criterion for the divergence of a sequence follows immediately from this result and the uniqueness of limits.

Corollary 3.51. If a sequence has subsequences that converge to different limits, then the sequence diverges.

Example 3.52. The sequence $((-1)^{n+1})$,

$$1,\ -1,\ 1,\ -1,\ 1,\ \dots,$$

has subsequences (1) and (-1) that converge to different limits, so it diverges.

In general, we define the limit set of a sequence to be the set of all limits of its convergent subsequences.

Definition 3.53. The limit set of a sequence (x_n) is the set

$$\{x \in \mathbb{R} : \text{there is a subsequence } (x_{n_k}) \text{ such that } x_{n_k} \to x \text{ as } k \to \infty\}$$

of limits of all of its convergent subsequences.

The limit set of a convergent sequence consists of a single point, namely its limit.

Example 3.54. The limit set of the divergent sequence $((-1)^{n+1})$,

$$1,\ -1,\ 1,\ -1,\ 1,\ \dots,$$

contains two points, and is $\{-1, 1\}$.

Example 3.55. Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0, 1]$. Every $x \in [0, 1]$ is a limit of a subsequence (r_{n_k}). To obtain such a subsequence recursively, choose $n_1 = 1$, and for each $k \geq 2$ choose a rational number r_{n_k} such that $|x - r_{n_k}| < 1/k$ and $n_k > n_{k-1}$. This is always possible since the rational numbers are dense in $[0, 1]$ and every interval contains infinitely many terms of the sequence. Conversely, if $r_{n_k} \to x$, then $0 \leq x \leq 1$ since $0 \leq r_{n_k} \leq 1$. Thus, the limit set of (r_n) is the interval $[0, 1]$.

Finally, we state a characterization of the lim sup and lim inf of a sequence in terms of of its limit set, where we use the usual conventions about $\pm\infty$. We leave the proof as an exercise.

Theorem 3.56. Suppose that (x_n) is sequence of real numbers with limit set S. Then

$$\limsup_{n\to\infty} x_n = \sup S, \qquad \liminf_{n\to\infty} x_n = \inf S.$$

3.9. The Bolzano-Weierstrass theorem

The Bolzano-Weierstrass theorem is a fundamental compactness result. It allows us to deduce the convergence of a subsequence from the boundedness of a sequence without having to know anything specific about the limit. In this respect, it is analogous to the result that a monotone increasing sequence converges if it is bounded from above, and it also provides another way of expressing the completeness of $\mathbb{R}$.

Theorem 3.57 (Bolzano-Weierstrass)**.** Every bounded sequence of real numbers has a convergent subsequence.

Proof. Suppose that (x_n) is a bounded sequence of real numbers. Let

$$M = \sup_{n\in\mathbb{N}} x_n, \qquad m = \inf_{n\in\mathbb{N}} x_n,$$

and define the closed interval $I_0 = [m, M]$.

Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where

$$L_0 = [m, (m + M)/2], \qquad R_0 = [(m + M)/2, M].$$

At least one of the intervals L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$ (even if the terms themselves are repeated).

Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals. One or both of the intervals L_1, R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such that $x_{n_2} \in I_2$. This is always possible because I_2 contains infinitely many terms of the sequence. Divide I_2 in half, pick a closed half-interval I_3 that contains infinitely many terms, and choose $n_3 > n_2$ such that $x_{n_3} \in I_3$. Continuing in this way, we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \dots I_k \supset \dots$ of length $|I_k| = 2^{-k}(M - m)$, together with a subsequence (x_{n_k}) such that $x_{n_k} \in I_k$.

Let $\epsilon > 0$ be given. Since $|I_k| \to 0$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $|I_k| < \epsilon$ for all $k > K$. Furthermore, since $x_{n_k} \in I_K$ for all $k > K$ we have $|x_{n_j} - x_{n_k}| < \epsilon$ for all $j, k > K$. This proves that (x_{n_k}) is a Cauchy sequence, and therefore it converges by Theorem 3.46. □

The subsequence obtained in the proof of this theorem is not unique. In particular, if the sequence does not converge, then for some $k \in \mathbb{N}$ both the left and right intervals L_k and R_k contain infinitely many terms of the sequence. In that case, we can obtain convergent subsequences with different limits, depending on our choice of L_k or R_k. This loss of uniqueness is a typical feature of compactness arguments.

We can, however, use the Bolzano-Weierstrass theorem to give a criterion for the convergence of a sequence in terms of the convergence of its subsequences. It states that if every convergent subsequence of a bounded sequence has the same limit, then the entire sequence converges to that limit.

Theorem 3.58. If (x_n) is a bounded sequence of real numbers such that every convergent subsequence has the same limit x, then (x_n) converges to x.

Proof. We will prove that if a bounded sequence (x_n) does not converge to x, then it has a convergent subsequence whose limit is not equal to x.

If (x_n) does not converges to x then there exists $\epsilon_0 > 0$ such that $|x_n - x| \geq \epsilon_0$ for infinitely many $n \in \mathbb{N}$. We can therefore find a subsequence (x_{n_k}) such that

$$|x_{n_k} - x| \geq \epsilon_0$$

for every $k \in \mathbb{N}$. The subsequence (x_{n_k}) is bounded, since (x_n) is bounded, so by the Bolzano-Weierstrass theorem, it has a convergent subsequence $(x_{n_{k_j}})$. If

$$\lim_{j \to \infty} x_{n_{k_j}} = y,$$

then $|x - y| \geq \epsilon_0$, so $x \neq y$, which proves the result. □